Exercise 4

Use residues to establish the following integration formula:

$$\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} = \frac{2\pi}{\sqrt{1 - a^2}} \quad (-1 < a < 1).$$

Solution

Because the integral goes from 0 to 2π and the integrand is in terms of $\sin\theta$, we can make the substitution, $z = e^{i\theta}$. Euler's formula states that $e^{i\theta} = \cos\theta + i\sin\theta$, so we can write $\cos\theta$ and $d\theta$ in terms of z and dz, respectively.

$$\cos\theta = \frac{z + z^{-1}}{2} \quad \text{and} \quad d\theta = \frac{dz}{iz}.$$

The integral becomes

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} &= \int_C \frac{1}{1 + a\left(\frac{z+z^{-1}}{2}\right)} \frac{dz}{iz} \\
&= \int_C \frac{2}{z^2 + \frac{2}{a}z + 1} \frac{dz}{ia} \\
&= \int_C \frac{2}{ia} \frac{dz}{(z - z_1)(z - z_2)} \\
&= \int_C f(z)\,dz,
\end{aligned}$$

where the contour C is the positively oriented unit circle centered at the origin and z_1 and z_2 are the zeros of $z^2 + \frac{2}{a}z + 1$.

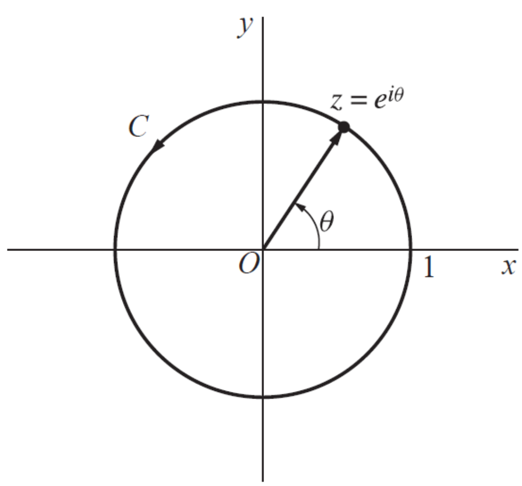

Figure 1: This figure illustrates the unit circle in the complex plane, where $z = x + iy$.

According to Cauchy's residue theorem, this contour integral is $2\pi i$ times the sum of the residues of $f(z)$ at the singular points inside the contour. That is,

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).$$

$f(z)$ has two singular points,

$$z_1 = \frac{-1-\sqrt{1-a^2}}{a}$$
$$z_2 = \frac{-1+\sqrt{1-a^2}}{a}.$$

Since $-1 < a < 1$, z_1 lies outside the unit circle, and it makes no contribution to the integral. However, z_2, does lie inside the circle, so we have to evaluate the residue of $f(z)$ at this point. Because z_2 is a simple pole, the residue can be written as

$$\operatorname*{Res}_{z=z_2} f(z) = \phi(z_2).$$

where $\phi(z)$ is determined from $f(z)$.

$$f(z) = \frac{\phi(z)}{z-z_2} \quad \rightarrow \quad \phi(z) = \frac{2}{ia}\frac{1}{z-z_1}$$

So

$$\operatorname*{Res}_{z=z_2} f(z) = \phi(z_2) = \frac{2}{ia}\frac{1}{z_2-z_1} = \frac{2}{ia}\frac{a}{2\sqrt{1-a^2}} = -\frac{i}{\sqrt{1-a^2}}$$

This means that

$$\int_C f(z)\,dz = 2\pi i\left(-\frac{i}{\sqrt{1-a^2}}\right) = \frac{2\pi}{\sqrt{1-a^2}}.$$

Therefore,

$$\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta} = \frac{2\pi}{\sqrt{1-a^2}} \quad (-1 < a < 1).$$